CONVERTING EULER ANGLES TO STARS ORIENTATION ANGLES

Nomenclature

α = STARS orientation angle `alpha`
β = STARS orientation angle `beta`
α_{eff} = effective aerodynamic angle of attack
β_{eff} = effective aerodynamic sideslip angle
ϕ = Euler roll angle
θ = Euler pitch angle
ψ = Euler yaw angle
B = coordinate transformation matrix from the body frame to global frame
C_a = shorthand notation for cos a
S_a = shorthand notation for sin a
$\mathbf{U}_g$ = free stream velocity vector in the global frame
$\mathbf{U}_b$ = free stream velocity vector in the body frame

Introduction

As shown in the figure below, STARS uses two orientation angles, `alpha` and `beta`, to define the direction of the free stream velocity vector with respect to the body frame. For simple cases, we typically think of `alpha` and `beta` as the angle of attack and sideslip angle respectively. However, this thinking will fail us for the general case of a body that is free to roll, pitch, or yaw in a manner described by the Euler angles ϕ, θ and ψ.

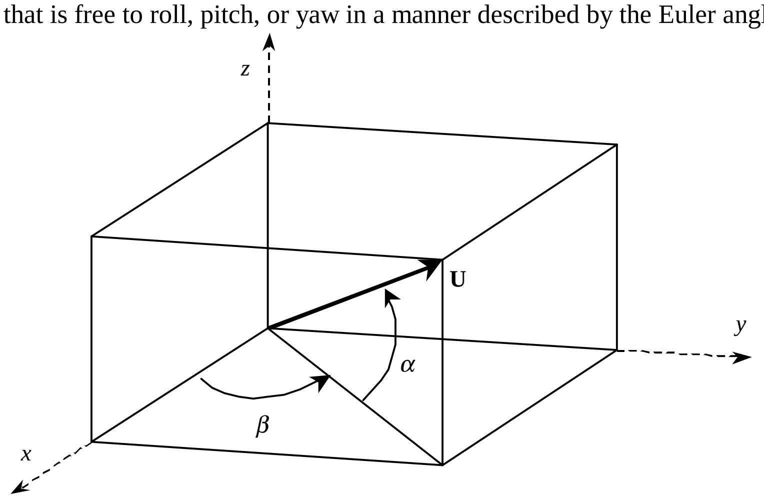

Figure 1: STARS orientation angles `alpha` and `beta`.

Notice in the above diagram that `alpha` and `beta` are identically equal to the Euler yaw and pitch angles for the case of zero roll, $\phi = 0$. This makes our choice of `alpha` and `beta` straightforward if our geometry is limited to zero roll. For the more general case with a nonzero roll angle, it is necessary to relate the geometric transformation described by the Euler angles to the similar transformation used by STARS and compute the appropriate values for `alpha` and `beta`.

Definition of free stream velocity vector in the body frame as used by STARS

Using simple trigonometry, we can define the free stream velocity vector in terms of the STARS orientation angles as follows:

(1)
$$\mathbf{U}_b = \begin{bmatrix} C_\alpha C_\beta \\ C_\alpha S_\beta \\ S_\alpha \end{bmatrix}$$

Derivation of free stream velocity vector in the body frame using Euler angles

Using the three Euler angles, a coordinate transformation matrix that relates vectors in the body frame to the global frame is defined as follows:

(2)
$$\mathbf{B} = \begin{bmatrix} C_\theta C_\psi & S_\phi S_\theta C_\psi - C_\phi S_\psi & C_\phi S_\theta C_\psi + S_\phi S_\psi \\ C_\theta S_\psi & S_\phi S_\theta S_\psi + C_\phi C_\psi & C_\phi S_\theta S_\psi - S_\phi C_\psi \\ -S_\theta & S_\phi C_\theta & C_\phi C_\theta \end{bmatrix}$$

The determinant of the above transformation matrix is identically one, and its inverse is formulated as follows:

(3)
$$\mathbf{B}^{-1} = \begin{bmatrix} C_\theta C_\psi & C_\theta S_\psi & -S_\theta \\ S_\phi S_\theta C_\psi - C_\phi S_\psi & S_\phi S_\theta S_\psi + C_\phi C_\psi & S_\phi C_\theta \\ C_\phi S_\theta C_\psi + S_\phi S_\psi & C_\phi S_\theta S_\psi - S_\phi C_\psi & C_\phi C_\theta \end{bmatrix}$$

The inverse transformation matrix defined above is used to convert a free stream velocity vector in the global frame to the body frame. In order to avoid negative values for the angles `alpha` and `beta`, we typically define our geometry such that the free stream velocity vector in the global frame points in the direction of the positive *x*-axis, *i.e.* $\mathbf{U}_g = \{\ 1.0,\ 0.0,\ 0.0\ \}^T$. This results in the following definition for the free stream velocity vector in terms of the three Euler angles:

(4)
$$\mathbf{U}_b = \mathbf{B}^{-1}\mathbf{U}_g = \begin{bmatrix} C_\theta C_\psi \\ S_\phi S_\theta C_\psi - C_\phi S_\psi \\ C_\phi S_\theta C_\psi + S_\phi S_\psi \end{bmatrix}$$

Note that our choice of free stream velocity vector is based on a coordinate system that is the inverse of the standard coordinate system used in flight dynamics, *i.e.* the *z*-axis points "upward," the *x*-axis points "backward," and the *y*-axis points out the "right-side."

Relationship between STARS orientation angles and Euler angles

Equating equations (1) and (4) and solving for α and β allows us to derive the following definitions for the STARS orientation angles in terms of the three Euler angles:

(5)
$$\alpha = \arcsin\left(C_\phi S_\theta C_\psi + S_\phi S_\psi\right)$$

(6)
$$\beta = \arcsin\left(\frac{S_\phi S_\theta C_\psi - C_\phi S_\psi}{C_\alpha}\right)$$

Effective aerodynamic angles

It is also interesting to compute the effective angle of attack and sideslip angle in terms of our two sets of orientation angles. Using the definition for the aerodynamic angle of attack, $\alpha_{\text{eff}} = \arctan(w/u)$, and sideslip angle, $\beta_{\text{eff}} = \arcsin(v/|\mathbf{U}|)$, we arrive at the following definition of these two aerodynamic angles:

(7)
$$\alpha_{\text{eff}} = \arctan\left(\frac{C_\phi S_\theta C_\psi + S_\phi S_\psi}{C_\theta C_\psi}\right) = \arctan\left(\frac{S_\alpha}{C_\alpha C_\beta}\right)$$

(8)
$$\beta_{\text{eff}} = \arcsin\left(S_\phi S_\theta C_\psi - C_\phi S_\psi\right) = \arcsin\left(C_\alpha S_\beta\right)$$

Notice that the orientation angle `alpha` is only equal to the aerodynamic angle of attack when $C_\beta = 1.0$, while the orientation angle `beta` is only equal to the aerodynamic sideslip angle when $C_\alpha = 1.0$.